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Thus, we have

$$F = 4mk^2\left(\frac{1}{d} + \cancel{\frac{\varepsilon\cos\theta}{d}} - \cancel{\frac{\varepsilon\cos\theta}{d}}\right)\cdot\frac{1}{r^2}$$

$$= \frac{4mk^2}{d}\cdot\frac{1}{r^2}$$

Which is an inverse square force.

Inverse Square Force $\Rightarrow$ Conic Trajectory

(1)

Newton's Law of Universal Gravitation

$$F = G\frac{Mm}{r^2}$$

G = gravitational constant
M = mass of object
m = mass of other object
r = distance between

Let's write $C = GM$ and choose the object with mass M to be at the origin.

Comparing this force with our previous formula, we get

$$F = \cancel{m}\left(\frac{4k^2}{r^3} - \frac{d^2r}{dt^2}\right) = C\frac{\cancel{m}}{r^2}$$

$$\frac{4k^2}{r^3} - \frac{d^2r}{dt^2} = \frac{C}{r^2} \quad (*)$$

Let $r = f(\theta)$ & $g(\theta) = \frac{1}{f(\theta)}$, so that

$$r = g(\theta)^{-1}$$

Then $$\frac{dr}{dt} = -(g(\theta))^{-2}\frac{d}{dt}(g(\theta)) = -(g(\theta))^{-2}g'(\theta)\frac{d\theta}{dt}$$

$$= -(g(\theta))^{-2}g'(\theta)\left(\frac{2k}{r^2}\right) = -\cancel{g(\theta)^{-2}}\,g'(\theta)(2k\cancel{g(\theta)^2})$$

$$= -2kg'(\theta)$$

and one more derivative

$$\frac{d^2r}{dt^2} = -2kg''(\theta)\cdot\frac{d\theta}{dt} = -2kg''(\theta)\cdot\frac{2k}{r^2} = -4k^2g''(\theta)\cdot\frac{1}{r^2}$$

Plugging this into $(*)$ gives

$$\left[\frac{4k^2}{r^3} + 4k^2 g''(\theta)\cdot\frac{1}{r^2} = \frac{C}{r^2}\right]\cdot\frac{r^2}{4k^2} \qquad (r \neq 0)$$

$$\Rightarrow \frac{1}{r} + g''(\theta) = \frac{C}{4k^2}$$

$$\Rightarrow g''(\theta) + g(\theta) = \frac{C}{4k^2}$$

Letting $g(\theta) = h(\theta) + \frac{C}{4k^2}$, we get

$$h''(\theta) + h(\theta) = 0 \iff h''(\theta) = -h(\theta)$$

The two functions which satisfy this are

$$\sin\theta \;\&\; \cos\theta$$

The most general solution is then

$$h(\theta) = A\sin\theta + B\cos\theta$$

$$\Rightarrow g(\theta) = A\sin\theta + B\cos\theta + \frac{C}{4k^2}$$

At this point, we can forget that θ is a function of t, since we're looking for the shape of the trajectory.

Unless other forces act on our particle, there will be a point of "closest approach" for P to the center of the force. This means that $f(\theta)$ has a local minimum, which means $g(\theta)$ has a local maximum. By rotating the axes if necessary, we can assume that this is at $\theta=0$. This means that $g'(0)=0$

$$g'(\theta) = A\cos\theta - B\sin\theta$$

$$g'(0) = A = 0$$

So, $g'(\theta) = -B\sin\theta \Rightarrow g''(\theta) = -B\cos\theta$

Now, if $B<0 \Rightarrow -B>0 \Rightarrow g''(0) = -B\cos 0 = -B > 0$

$\Rightarrow g$ has a local minimum at $\theta=0$, a contradiction.

Thus $B \geq 0$

$$\Rightarrow g(\theta) = B\cos\theta + \frac{C}{4k^2}$$

$$\Rightarrow f(\theta) = \frac{1}{B\cos\theta + \frac{C}{4k^2}} = \frac{\frac{4k^2}{C}}{1 + \frac{4k^2 B}{C}\cos\theta}$$

So, $d = \frac{4k^2}{C}$ & $\varepsilon = \frac{4k^2 B}{C}$

fitting this with $\boxed{r = f(\theta) = \frac{d}{1+\varepsilon\cos\theta}}$ $\Rightarrow$ Kepler's First Law

(Only closed orbits are ellipses)

$\boxed{\text{Elliptical orbit}}$

$L = 2d$ is the latus rectum.

a = semimajor axis b = semiminor axis

T = period of the orbit

$\Rightarrow k = \frac{\pi ab}{T}$ & $L = \frac{2b^2}{a}$

$\Rightarrow C = \frac{4k^2}{d} = \frac{8k^2}{L}$

$\Rightarrow F = \frac{Cm}{r^2} = \frac{8mk^2}{L} \cdot \frac{1}{r^2} = \cancel{8}^{4} m \cdot \frac{a}{\cancel{2}\cancel{b^2}} \cdot \frac{\pi^2 a^2 \cancel{b^2}}{T^2} \frac{1}{r^2}$

$\Rightarrow F = \frac{4\cancel{m}\pi^2 a^3}{T^2} \frac{1}{\cancel{r^2}} = G\cancel{m}M \cdot \frac{1}{\cancel{r^2}}$

$\Rightarrow \frac{4\pi^2 a^3}{T^2} = GM \iff \boxed{\frac{a^3}{T^2} = \frac{GM}{4\pi^2}}$ Kepler's 3rd Law.